NORTH SYDNEY GIRLS HIGH SCHOOL

HSC 2015 Extension 1 Mathematics Assessment Task 1
Term 4, 2014

Name:________________________________ **Mathematics Class: 11Mx**_______

Student Number: ____________________________

Time Allowed: **55 minutes + 2 minutes reading time**

Total Marks: **41**

Instructions:

- Attempt all questions.
- Start each question in a new booklet.
  Put your <u>number</u> on every booklet and any extra writing paper used.
- Show all necessary working.
- Marks may be deducted for incomplete or poorly arranged work.
- Work down the page.
- Do not work in columns.
- Each question will be collected separately. Submit a blank booklet if you do not attempt a question.

Question	1 – 3	4 – 5	6a	6b	7a	7b	8a	8bc	Total
PE3	/3		/4			/4	/3	/9	/23
PE4				/8					/8
HE7		/2			/8				/10
	/3	/2	/4	/8	/8	/4	/3	/9	/41

Section I

5 marks

Attempt Questions 1 - 5

Use the multiple choice answer sheet for Questions $1-5$

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1 When $2x^3 + x^2 + kx - 4$ is divided by $(x - 1)$ the remainder is 2.

What is the value of k?

(A) -7

(B) -5

(C) 1

(D) 3

2 A function is represented by the parametric equations

$$x = 2t + 1$$
$$y = t - 2$$

Which of the following is the Cartesian equation of the function?

(A) $x - 2y + 3 = 0$

(B) $x - 2y - 3 = 0$

(C) $x + 2y + 5 = 0$

(D) $x - 2y - 5 = 0$

3 The polynomial $P(x)$ is monic and of degree 5.
It has a single zero at $x = -1$ and a double zero at $x = 2$.

The other two zeroes are not real.
Which of the following equations best represents $P(x)$?

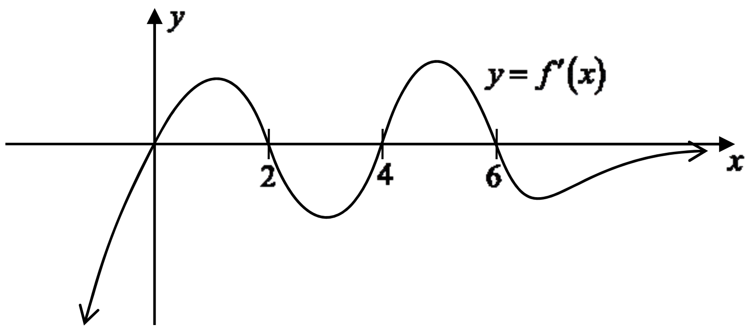
(A) $(x - 1)(x + 2)^2(x^2 + bx + c)$, where $b^2 - 4c > 0$

(B) $(x + 1)(x - 2)^2(x^2 + bx + c)$, where $b^2 - 4c > 0$

(C) $(x + 1)(x - 2)^2(x^2 + bx + c)$, where $b^2 - 4c < 0$

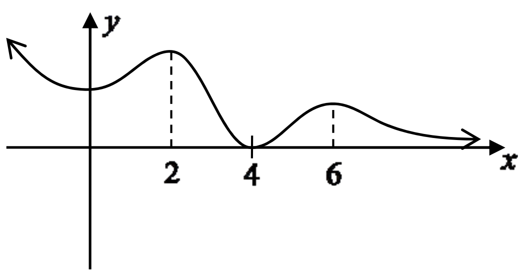
(D) $(x - 1)(x + 2)^2(x^2 + bx + c)$, where $b^2 - 4c < 0$

4 Following is the sketch of $y = f'(x)$, where $f'(x)$ is the derivative of the function $f(x)$.



Which of the following graphs is a possible graph of the original function $y = f(x)$?

(A)



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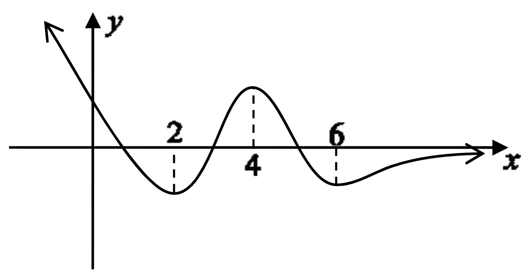
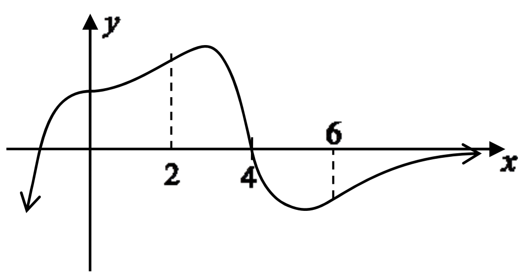
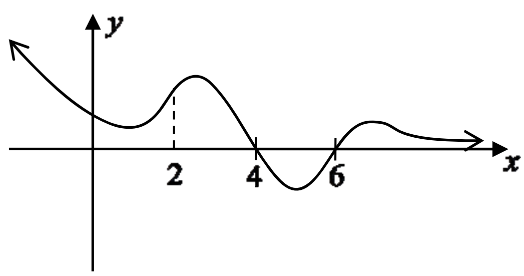


(C)

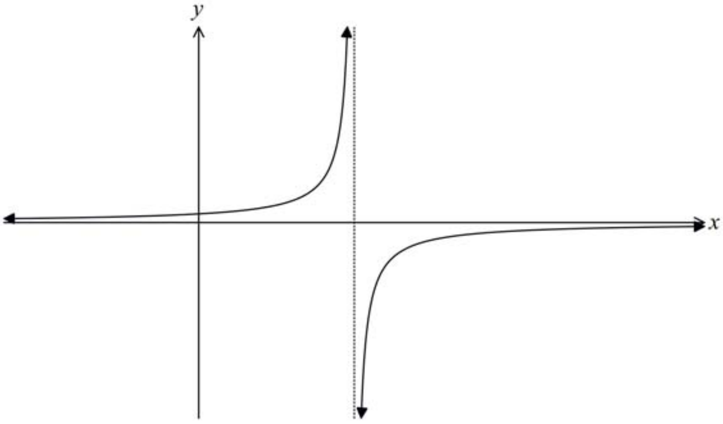


(D)



5 Which of the following best represents the graph of $y = \dfrac{1}{\sqrt{4 - x^2}}$?

(A)



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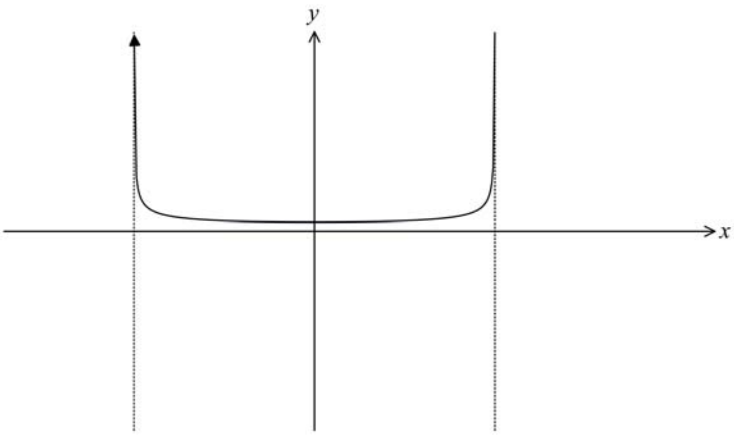

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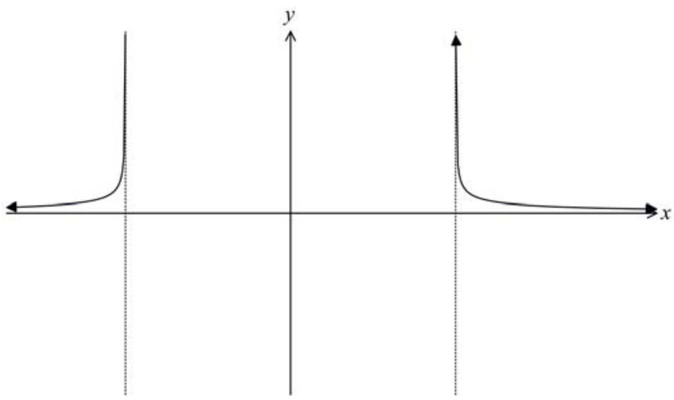
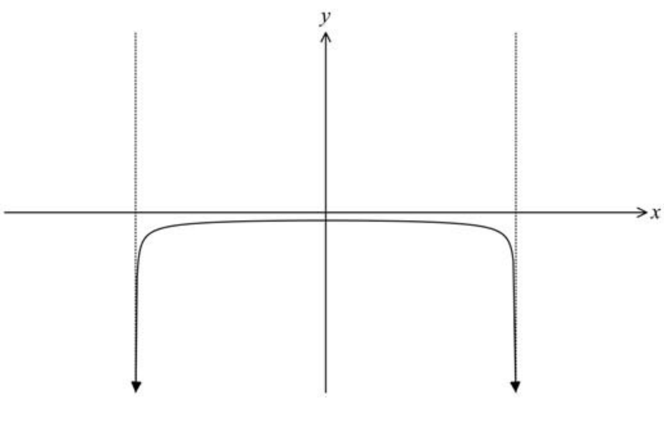


(D)



Section II

36 marks
Attempt Questions 6–8

Answer each question in a SEPARATE writing booklet. Extra writing paper is available.

In Questions 6–8, your responses should include relevant mathematical reasoning and/or calculations.

Question 6 (12 marks) Use a SEPARATE writing booklet.

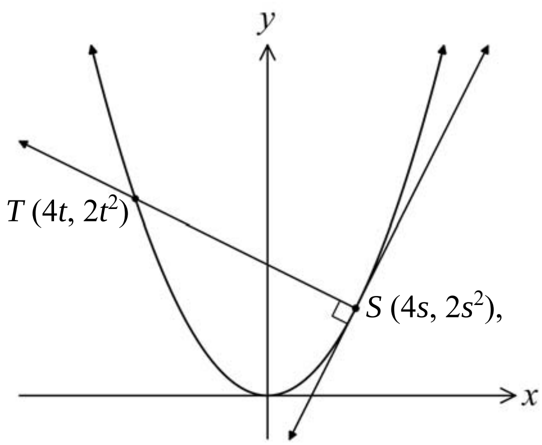
(a) Given that α, β and γ are the roots of the equation $x^3 - 4x^2 + 3x - 1 = 0$, find the value of:

(i) $\alpha + \beta + \gamma$ **1**

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ **1**

(iii) $\alpha^2 + \beta^2 + \gamma^2$ **2**

(b) The point $S\left(4s, 2s^2\right)$, lies on the parabola as shown below.



The normal at S intersects the parabola again at the point $T\left(4t, 2t^2\right)$.

(i) Write down the Cartesian equation of the parabola. **1**

(ii) By finding the equation of the normal, show that ST passes through the point $N\left(0, 4 + 2s^2\right)$. **3**

(iii) By finding the equation of the chord ST, show that $t = -\left(s + \dfrac{2}{s}\right)$. **3**

(iv) With reference to the diagram, explain why $s \neq 0$. **1**

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) Consider the cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c and d are real numbers and $a \neq 0$.

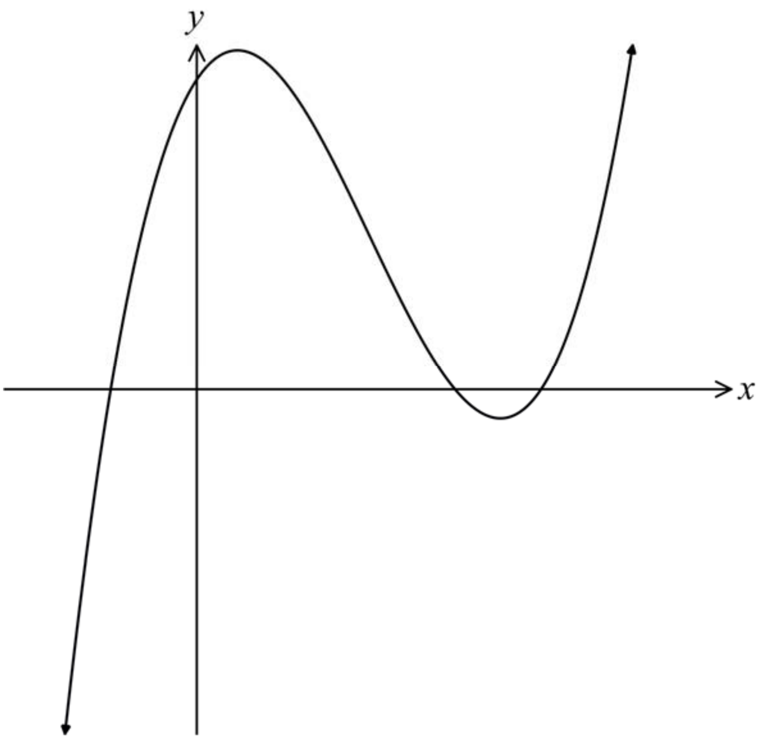
Let α, β and γ be the zeroes of $f(x)$.

(i) Explain why all cubic polynomial functions have a single point of inflexion where the second derivative is zero. **2**

(ii) Using part (i) above, show that the x-coordinate of the point of inflexion on the curve $y = f(x)$ is given by **3**

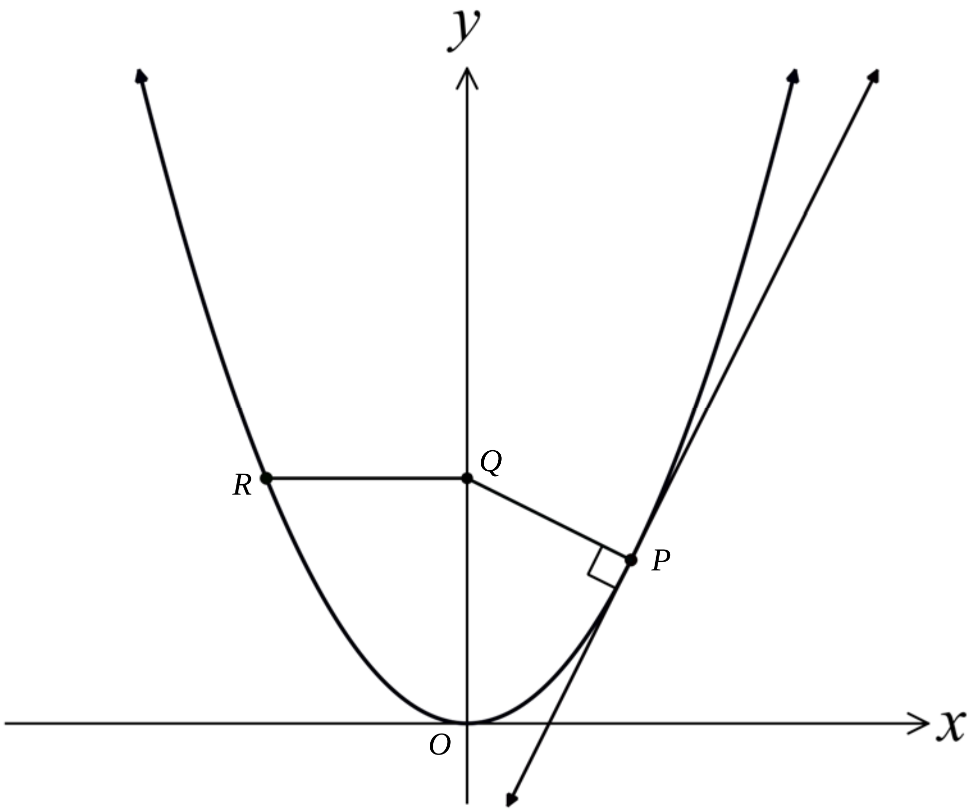
$$x = \frac{\alpha + \beta + \gamma}{3}$$

(iii) The cubic polynomial below has x-intercepts at -1, 3 and 4. **3**
If the y-intercept is 24, find the coordinates of the point of inflexion.



Question 7 (continued)

(b)



The diagram above shows the graph of the parabola $x^2 = 4ay$.

The normal to the parabola at the variable point $P\left(2at, at^2\right)$, $t > 0$, cuts the y-axis at Q. Point R lies on the parabola.

You may assume that the equation of the normal to the parabola at P is given by $x + ty = 2at + at^3$ (Do NOT prove.)

(i) The point R is such that QR is parallel to the x-axis and $\angle PQR > 90°$. **2**
Show that the coordinates of R are $\left(-2a\sqrt{t^2+2}, at^2+2a\right)$.

(ii) Let M be the midpoint of RQ. Find the Cartesian equation of the locus of M. **2**
(Do NOT consider any possible domain restrictions of the locus)

Question 8 (12 marks) Use a SEPARATE writing booklet.

(a) The polynomial $P(x) = x^3 + ax^2 + bx + 20$ has a factor of $x - 5$ and leaves a remainder of -10 when divided by $x - 3$. **3**

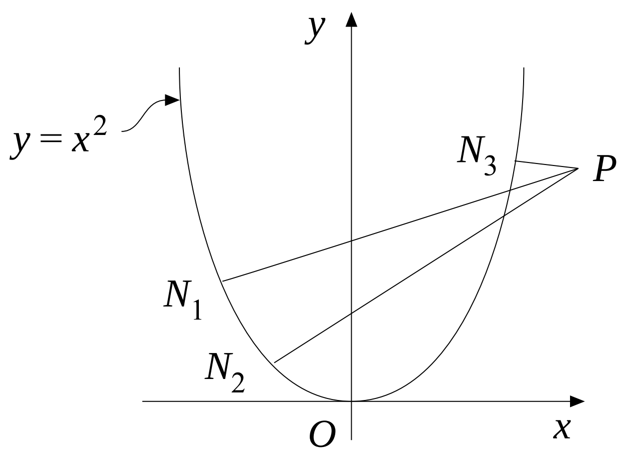
Find the values of a and b.

(b) A curve has parametric equations $x = \frac{1}{t} - 1$ and $y = 2t + \frac{1}{t^2}$

(i) Find $\frac{dy}{dx}$ in terms of t. **2**

(ii) Find the coordinates of any stationary points and determine their nature. **2**

(c) Consider the diagram below of the parabola $y = x^2$.
Some points (e.g P) lie on three distinct normals (PN_1, PN_2 and PN_3) to the parabola.



(i) Show that the equation of the normal to $y = x^2$ at the point (t, t^2) may be written as **2**

$$t^3 + \left(\frac{1-2y}{2}\right)t + \left(\frac{-x}{2}\right) = 0$$

(ii) For polynomials of the form $p(x) = x^3 + cx + d$, it is known that if the polynomial has 3 distinct roots then $27d^2 + 4c^3 < 0$ (Do NOT prove this.) **3**

Suppose that the normal to $y = x^2$ at three distinct points $N_1(t_1, t_1^2)$, $N_2(t_2, t_2^2)$ and $N_3(t_3, t_3^2)$ all pass through $P(x_0, y_0)$.

Show that the coordinates of P satisfy $y_0 > 3\left(\frac{x_0}{4}\right)^{\frac{2}{3}} + \frac{1}{2}$

End of paper

(b) A curve has equation $y = \dfrac{x^2}{(x-1)(x-5)}$.

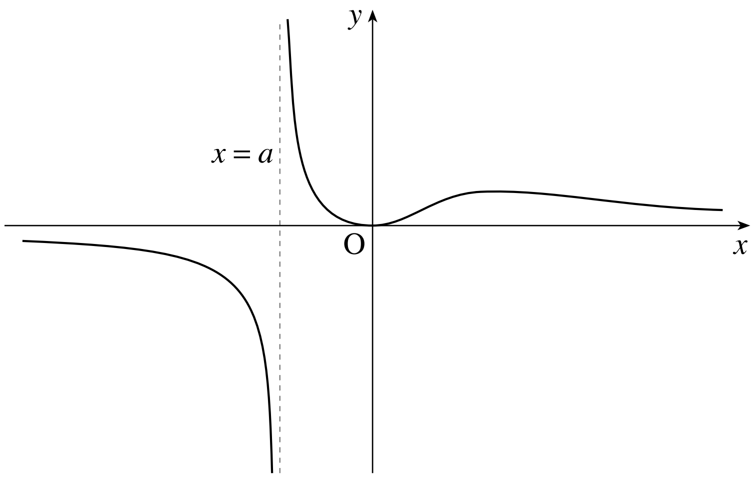
(i) By considering when the curve intersects with the line $y = k$, show that the stationary points of the curve satisfy $k(4k + 5) = 0$. **2**

(ii) Write down the coordinates of the stationary points on the curve. **2**

(ii) Sketch the curve showing intercepts, asymptotes and stationary points. **3**

(a)

Fig. 7 shows the curve $y = \dfrac{x^2}{1 + 2x^3}$. It is undefined at $x = a$; the line $x = a$ is a vertical asymptote.



(i) Write down the value of a correct to 3 sig fig. **1**

(ii) Show that $\dfrac{dy}{dx} = \dfrac{2x - 2x^4}{(1 + 2x^3)^2}$ **2**

(iii) For what values of k, where k is a constant, does the equation $2kx^3 - x^2 + k = 0$ have 3 distinct roots. **2**

(b) Consider the curve $y = x^3 - 4x$

(i) Show that the gradient of the tangent to the curve at the point $P(p, p^3 - 4p)$ is $3p^2 - 4$ **1**

(ii) The tangent at P cuts the curve again at the point R. Find the coordinates of R. 2/3

(b) Find the value of a, given that

$$x^3 - 4x^2 + a \equiv (x + 1)Q(x) + 3, \text{ where } Q(x) \text{ is a polynomial.}$$

It is given that α, β and γ satisfy the equations

$$\begin{aligned} \alpha + \beta + \gamma &= 1 \\ \alpha^2 + \beta^2 + \gamma^2 &= -5 \\ \alpha^3 + \beta^3 + \gamma^3 &= -23 \end{aligned}$$

(a) Show that $\alpha\beta + \beta\gamma + \gamma\alpha = 3$. *(3 marks)*

(b) Use the identity

$$(\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha) = \alpha^3 + \beta^3 + \gamma^3 - 3\alpha\beta\gamma$$

to find the value of $\alpha\beta\gamma$. *(2 marks)*

(c) Write down a cubic equation, with integer coefficients, whose roots are α, β and γ. *(2 marks)*

(d) Explain why this cubic equation has two non-real roots. *(2 marks)*

(e) Given that α is real, find the values of α, β and γ. *(4 marks)*

A curve is defined by the parametric equations $x = 2t + \frac{1}{t^2}$, $y = 2t - \frac{1}{t^2}$.

(a) At the point P on the curve, $t = \frac{1}{2}$.

(i) Find the coordinates of P. *(2 marks)*

(ii) Find an equation of the tangent to the curve at P. *(5 marks)*

(b) Show that the cartesian equation of the curve can be written as

$$(x - y)(x + y)^2 = k$$

where k is an integer. *(3 marks)*

Find the value of a, given that $x^3 - 4x^2 + a \equiv (x + 1)Q(x) + 3$, where $Q(x)$ is a polynomial **2**

2 A curve is defined by the parametric equations

$$x = \frac{1}{t}, \qquad y = t + \frac{1}{2t}$$

(a) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ in terms of t.

(b) Find an equation of the normal to the curve at the point where $t = 1$.

(c) Show that the cartesian equation of the curve can be written in the form

$$x^2 - 2xy + k = 0$$

where k is an integer.

Spare Questions

6 Suppose $P(x) = 2x^3 + 5x^2 + 2x + 9$, $Q(x) = x^3 + 5x^2 + 5x + 7$ and $P(x) - Q(x) = (x - 1)^2\,(x + 2)$.

What is the geometric interpretation of this?

(A) $P(x)$ and $Q(x)$ are tangent at $x = 1$ and intersect at $x = -2$.

(B) $P(x)$ and $Q(x)$ are intersect at $x = 1$ and tangent at $x = -2$.

(C) $P(x)$ and $Q(x)$ are intersect at both $x = 1$ and $x = -2$.

(D) $P(x)$ and $Q(x)$ are tangent at both $x = 1$ and $x = -2$.

4 The equation $x^3 - 2x^2 - x + 1 = 0$ has roots α, β and γ.
Which of the following is true?

(A) $\alpha + \beta + \gamma = -2$ and $\alpha\beta\gamma = -1$

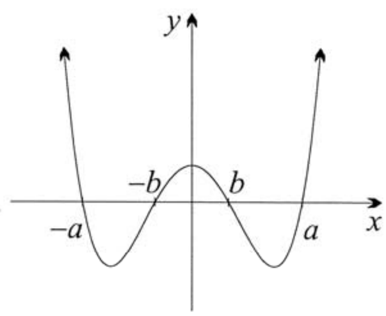
(B) $\alpha + \beta + \gamma = -2$ and $\alpha\beta\gamma = 1$

(C) $\alpha + \beta + \gamma = 2$ and $\alpha\beta\gamma = -1$

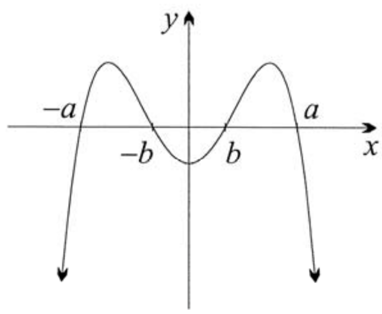
(D) $\alpha + \beta + \gamma = 2$ and $\alpha\beta\gamma = 1$

7 Which diagram best represents $P(x) = (x - a)^2(b^2 - x^2)$, where $a > b$?

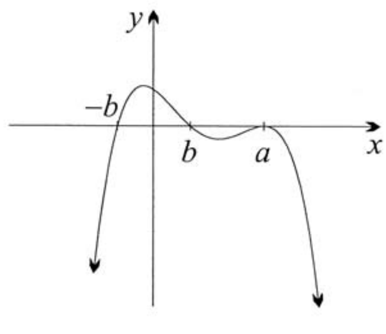
(A)



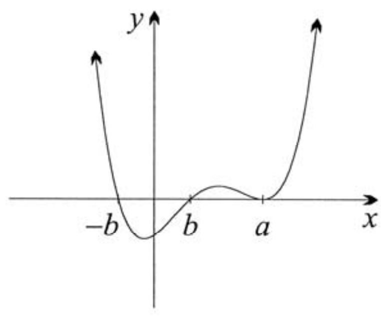
(B)



(C)



(D)



8 If $x + \alpha$ is a factor of $7x^3 + 9x^2 - 5\alpha x$, where $a \neq 0$, what is the value of α?

(A) 2

(B) $\dfrac{4}{7}$

(C) $-\dfrac{4}{7}$

(D) -2

9 A polynomial of degree four is divided by a polynomial of degree two.
What is the maximum possible degree of the remainder?

(A) 3

(B) 2

(C) 1

(D) 0

10 It is known that $(x + 2)$ is a factor of the polynomial $P(x)$ and that

$$P(x) = (x^2 + x + 1) \times Q(x) + (2x + 3)$$

for some polynomial $Q(x)$.

From this information alone, which of the following can be deduced?

(A) $Q(-2) = -\frac{1}{3}$

(B) $Q(-2) = \frac{1}{3}$

(C) $Q(2) = -1$

(D) $Q(2) = 1$

3 What is the x-intercept of the normal to the parabola $x^2 = 4ay$ at the point $(2ap, ap^2)$?

(A) $ap(p^2 + 1)$

(B) $ap(p^2 + 2)$

(C) ap^2

(D) $-ap^2$

Given that $x(2x - 1)(x + 1) + 3 \equiv 2x^3 + bx^2 + cx + 3$, find the values of b and c. **2**

The cubic equation

$$x^3 + px^2 + qx + r = 0$$

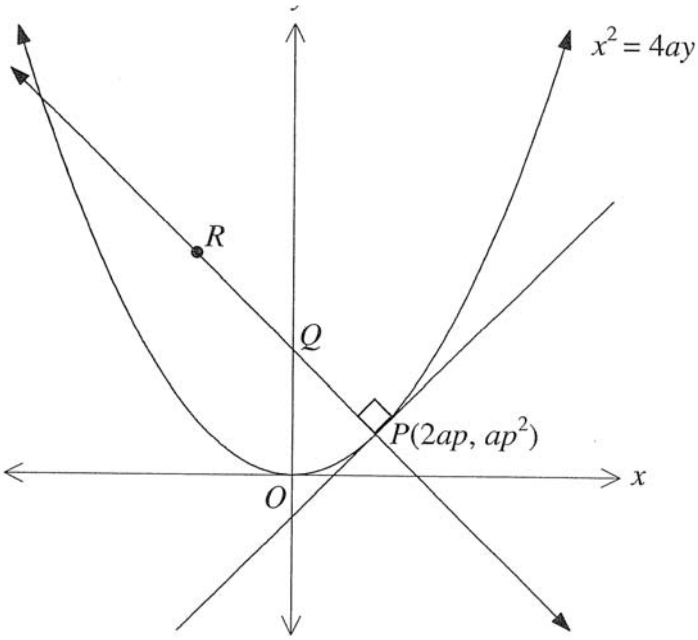
where p, q and r are real, has roots α, β and γ.

(a) Given that

$$\alpha + \beta + \gamma = 4 \qquad \text{and} \qquad \alpha^2 + \beta^2 + \gamma^2 = 20$$

find the values of p and q.

(b) Given further that one root is $3 + \mathrm{i}$, find the value of r.



The diagram shows a variable point $P\left(2ap, ap^2\right)$ on the parabola $x^2 = 4ay$.

The normal to the parabola at P intersects the y-axis at Q. The point Q is the midpoint of PR.

The equation of the normal is $x + py - 2ap - ap^3 = 0$. (Do NOT prove this.)

(i) Find the coordinates of the point Q. **1**

(ii) The locus of the point R is a parabola. **3**

Find the equation of this parabola in Cartesian form and state its vertex.

(a) When a polynomial $P(x)$ is divided by $(x - 1)$, the remainder is 3. When $P(x)$ is divided by $(x + 2)$, the remainder is -2. Find the remainder when the polynomial is divided by $x^2 + x - 2$. **2**

(b) The tangent to the curve $y = x^3 - 4x^2 - x + 2$, at a point Q on the curve, intersects the curve again at $A(2, -8)$. Find the co-ordinates of the point Q. **3**

(i) Show that 1 is a zero of $P(x)$. **1**

(ii) Express $P(x)$ as a product of three factors. **3**

(iii) Sketch the graph of $y = P(x)$. Show clearly all the intercepts with axes. Do not calculate the coordinates of the turning points. **1**

(iv) Solve the inequality $P(x) \leq 0$. **1**

(b) The displacement of a particle moving in simple harmonic motion is given by

$$x = a \cos nt,$$

QUESTION THREE (14 marks) Use a separate writing booklet. **Marks**

(a) Let the roots of the equation $x^3 + 2x^2 - 3x + 5 = 0$ be α, β and γ.

(i) State the values of:

NORTH SYDNEY GIRLS HIGH SCHOOL

HSC 2015 Extension 1 Mathematics Assessment Task 1

Term 4, 2014

Sample Solutions

Section I

5 marks
Attempt Questions 1 - 5
Use the multiple choice answer sheet for Questions 1 – 5

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1 When $2x^3 + x^2 + kx - 4$ is divided by $(x - 1)$ the remainder is 2.

What is the value of k?

(A) -7

(B) -5

(C) 1

(D) 3

Let $P(x) = 2x^3 + x^2 + kx - 4$
$\therefore P(1) = 2$
$\therefore 2 + 1 + k - 4 = 2 \Rightarrow k = 3$

2 A function is represented by the parametric equations

$$x = 2t + 1$$
$$y = t - 2$$

Which of the following is the Cartesian equation of the function?

(A) $x - 2y + 3 = 0$

(B) $x - 2y - 3 = 0$

(C) $x + 2y + 5 = 0$

(D) $x - 2y - 5 = 0$

$t = y + 2$
$\therefore x = 2(y + 2) + 1$
$\therefore x = 2y + 5$
$\therefore x - 2y - 5 = 0$

3 The polynomial $P(x)$ is monic and of degree 5.
It has a single zero at $x = -1$ and a double zero at $x = 2$.

The other two zeroes are not real.
Which of the following equations best represents $P(x)$?

(A) $(x - 1)(x + 2)^2(x^2 + bx + c)$, where $b^2 - 4c > 0$

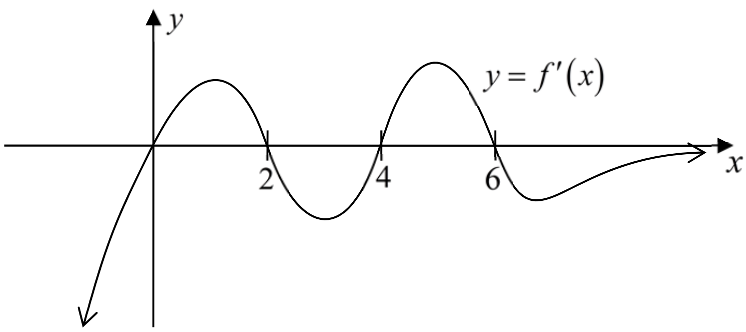
(B) $(x + 1)(x - 2)^2(x^2 + bx + c)$, where $b^2 - 4c > 0$

(C) $(x + 1)(x - 2)^2(x^2 + bx + c)$, where $b^2 - 4c < 0$

(D) $(x - 1)(x + 2)^2(x^2 + bx + c)$, where $b^2 - 4c < 0$

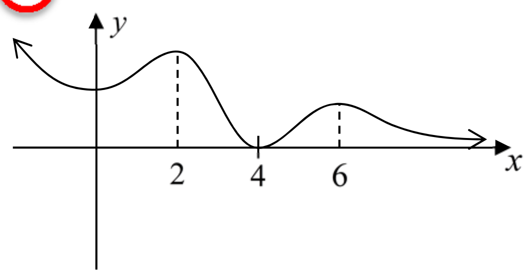
A single root at $x = -1$ and a double zero at $x = 2$ means $(x + 1)(x - 2)^2$
The other two zeroes are not real means that for $x^2 + bx + c)$ then $b^2 - 4c < 0$.

4 Following is the sketch of $y = f'(x)$, where $f'(x)$ is the derivative of the function $f(x)$.



Which of the following graphs is a possible graph of the original function $y = f(x)$?

(A)



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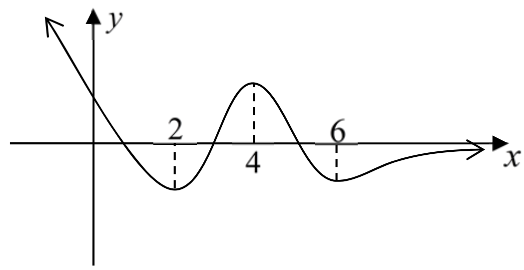
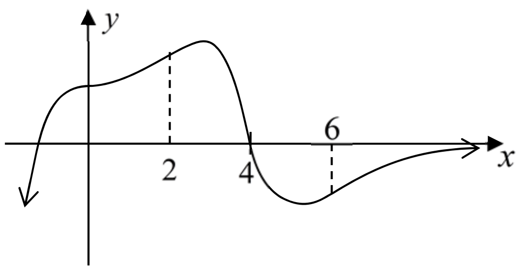
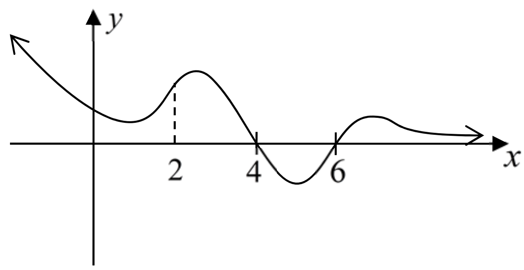


(C)



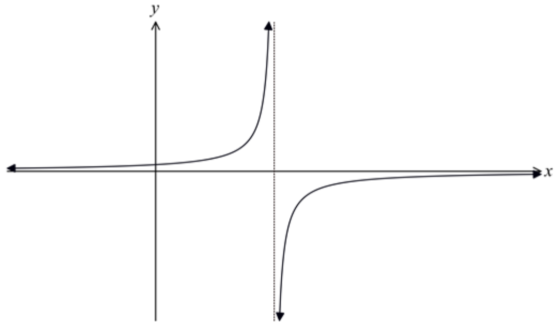
(D)



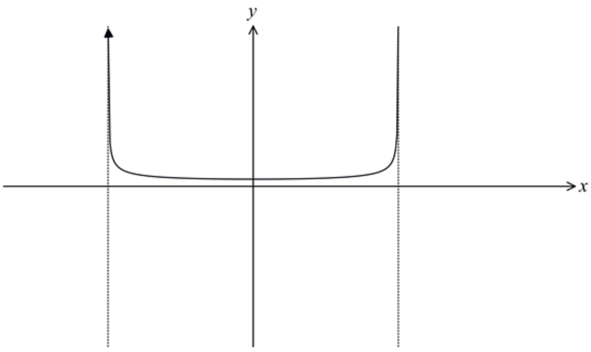
Stationary points are at $x = 2$, 4 and 6.
Looking at the sign of $f'(x)$ either side of $x = 2$, 4 and 6 means that there is maxima at $x = 2$, 6 and a minimum at $x = 4$.

5 Which of the following best represents the graph of $y = \dfrac{1}{\sqrt{4-x^2}}$?

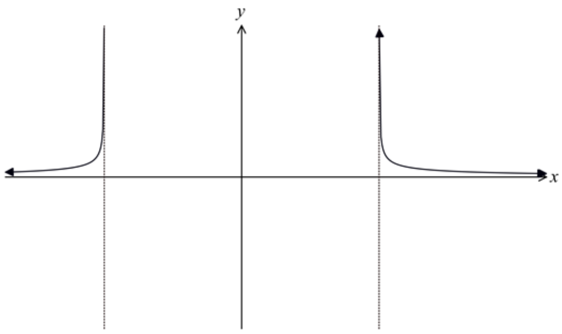
(A)



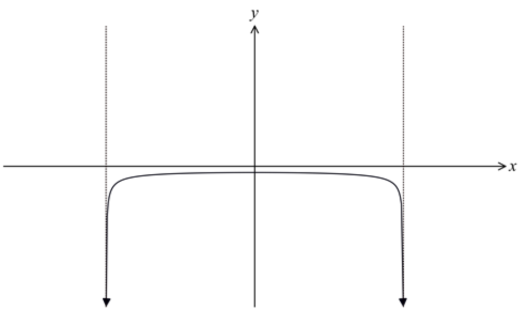
(B)



(C)



(D)



Considering range: $y > 0$

Considering domain: $4 - x^2 > 0 \Rightarrow -2 < x < 2$

Section II

Question 6 (12 marks)

(a) Given that α, β and γ are the roots of the equation $x^3 - 4x^2 + 3x - 1 = 0$, find the value of:

(i) $\alpha + \beta + \gamma$ **1**

$\alpha + \beta + \gamma = -(-4) = 4$

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} + \dfrac{1}{\gamma}$ **1**

$$\begin{aligned}\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} &= \frac{\sum \alpha\beta}{\alpha\beta\gamma} \\ &= \frac{3}{-(-1)} \\ &= 3\end{aligned}$$

(iii) $\alpha^2 + \beta^2 + \gamma^2$ **2**

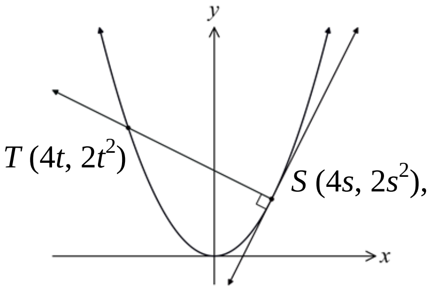
$$\begin{aligned}\alpha^2 + \beta^2 + \gamma^2 &= \left(\sum \alpha\right)^2 - 2\left(\sum \alpha\beta\right) \\ &= (-4)^2 - 2 \times 3 \\ &= 10\end{aligned}$$

Markers Comments

Generally well done, though the common errors were associated with incorrectly remembering $\alpha^2 + \beta^2 + \gamma^2 = \left(\sum \alpha\right)^2 - 2\left(\sum \alpha\beta\right)$

Question 6 (continued)

(b) The point $S\left(4s,\,2s^2\right)$, lies on the parabola as shown below.



The normal at S intersects the parabola again at the point $T\left(4t,\,2t^2\right)$.

(i) Write down the Cartesian equation of the parabola. **1**

$$x = 4s,\, y = \underbrace{2}_{a}s^2 \Rightarrow x^2 = 4\times 2y$$

$$\therefore x^2 = 8y$$

Markers Comments

Generally well done

(ii) By finding the equation of the normal, show that ST passes through the point $N\left(0,\,4+2s^2\right)$. **3**

$$x = 4s \Rightarrow \frac{dx}{ds} = 4$$

$$y = 2s^2 \Rightarrow \frac{dy}{ds} = 4s$$

$$\frac{dy}{dx} = \frac{\frac{dy}{ds}}{\frac{dx}{ds}}$$

$$= \frac{4s}{4}$$

$$= s$$

$\therefore$ the gradient of the normal is $-\frac{1}{s}$

$\therefore$ the normal is $y - 2s^2 = -\frac{1}{s}\left(x-4s\right)$

$\therefore x + sy = 4s + 2s^3$

Considering the y-intercept i.e. x = 0 then $sy = 4s + 2s^3$

$\therefore$ the normal passes through $N\left(0,\,4+2s^2\right)$

Markers Comments

Generally well done

Question 6 (continued)

(iii) By finding the equation of the chord ST, show that $t = -\left(s + \frac{2}{s}\right)$. **3**

$$m_{ST} = \frac{2t^2 - 2s^2}{4t - 4s}$$
$$= \frac{2(t-s)(t+s)}{4(t-s)}$$
$$= \frac{t+s}{2}$$

$\therefore$ chord ST is $y - 2t^2 = \left(\frac{s+t}{2}\right)(x - 4t)$

$\therefore$ the gradient of NT is also $\frac{s+t}{2}$

$$m_{NT} = \frac{4 + 2s^2 - 2t^2}{0 - 4t}$$
$$= -\frac{2 + s^2 - t^2}{2t}$$

$$\therefore \frac{t+s}{2} = -\frac{2 + s^2 - t^2}{2t}$$
$$\therefore t(t+s) = -\left(2 + s^2 - t^2\right)$$
$$\therefore ts = -2 - s^2$$
$$\therefore t = \frac{-2 - s^2}{s} = -\left(s + \frac{2}{s}\right)$$

Markers Comments

- Generally well done
- Some students could not complete the question as they did not realise that $m_{NT} = m_{ST}$
- Most students substituted $N\left(0, 4 + 2s^2\right)$ into chord ST.
- Some students used $m_{NT} = -\frac{1}{s}$

(iv) With reference to the diagram, explain why $s \neq 0$. **1**

At the origin the normal does not re-intersect the parabola and so $s \neq 0$

Markers Comments

- Some students explained the answer algebraically without reference to the diagram.
- Some students did not understand what the question meant for them to do.

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) Consider the cubic polynomial $f(x) = ax^3 + bx^2 + cx + d$ where a, b, c and d are real numbers and $a \neq 0$.

Let α, β and γ be the zeroes of $f(x)$.

(i) Explain why all cubic polynomial functions have a single point of inflexion where the second derivative is zero. **2**

$$f(x) = ax^3 + bx^2 + cx + d$$
$$\therefore f'(x) = 3ax^2 + 2bx + c$$
$$\therefore f''(x) = 6ax + 2b$$
$$= 2(3ax + b)$$

So there is only one POSSIBLE point of inflexion (POI).
Why is it a POI?
Consider the graph of $y = 2(3ax + b)$:

The straight line cuts the x-axis at $x = -\frac{b}{3a}$ and so either side of this point the graph changes sign i.e. $f(x)$ changes in concavity.

$\therefore$ there is only one POI at $x = -\frac{b}{3a}$.

Markers Comments

Most students did not consider that the 2nd derivative must change sign at $x = -\frac{b}{3a}$.

(ii) Using part (i) above, show that the x-coordinate of the point of inflexion on the curve $y = f(x)$ is given by **3**

$$x = \frac{\alpha + \beta + \gamma}{3}$$

From (ii) above the point of inflexion is at $x = -\frac{b}{3a}$.

$$\alpha + \beta + \gamma = -\frac{b}{a}$$
$$\therefore \frac{\alpha + \beta + \gamma}{3} = -\frac{b}{3a}$$

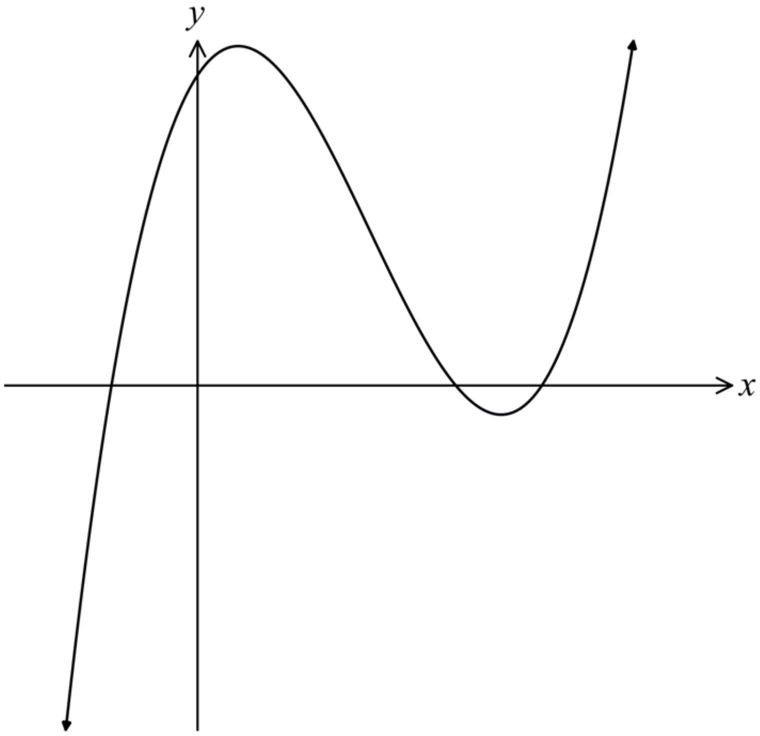
$\therefore$ the POI is at $x = \frac{\alpha + \beta + \gamma}{3}$.

Markers Comments

The most common error was to let $a = 1$.

Question 7 (continued)

(a) (iii) The cubic polynomial below has x-intercepts at –1, 3 and 4. **3**
If the y-intercept is 24, find the coordinates of the point of inflexion.



The equation of the polynomial is $y = k(x+1)(x-3)(x-4)$

With the y-intercept of 24 then $y = 2(x+1)(x-3)(x-4)$

From (iii), the x-coordinate of the POI is $x = \dfrac{-1+3+4}{3} = 2$

So the y-coordinate is given by $y = 2(2+1)(2-3)(2-4) = 12$

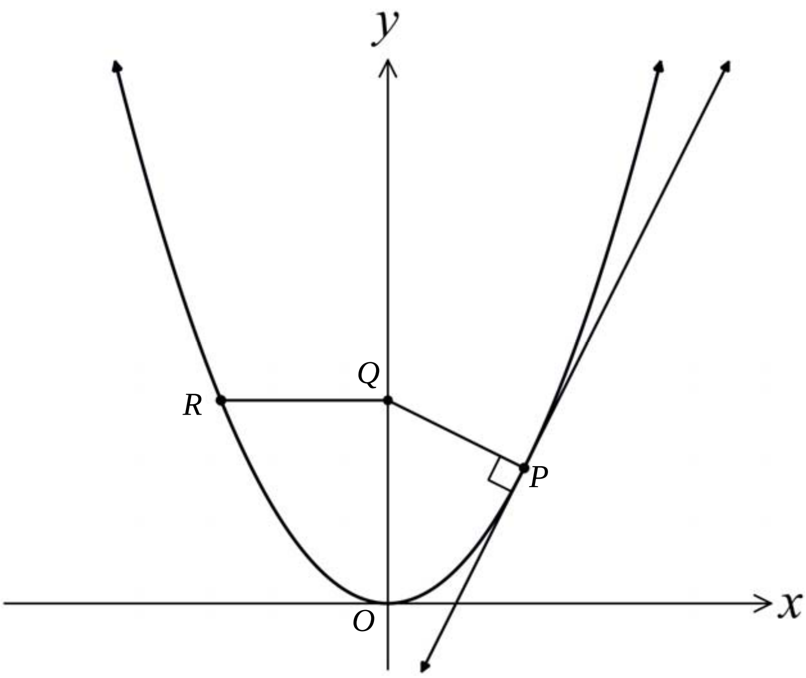
$\therefore$ the POI is at (2, 12).

Markers Comments

Students who used $y = ax^3 + bx^2 + cx + d$ in stead of $y = a(x+1)(x-3)(x-4)$ struggled to complete this question accurately.

Question 7 (continued)

(b)



The diagram above shows the graph of the parabola $x^2 = 4ay$.

The normal to the parabola at the variable point $P\left(2at, at^2\right)$, $t > 0$, cuts the y-axis at Q. Point R lies on the parabola.

You may assume that the equation of the normal to the parabola at P is given by $x + ty = 2at + at^3$ (Do NOT prove.)

(i) The point R is such that QR is parallel to the x-axis and $\angle PQR > 90°$. **2**
Show that the coordinates of R are $\left(-2a\sqrt{t^2+2}, at^2+2a\right)$.

The y-coordinate of R is the same as that of Q:
With $x + ty = 2at + at^3$, let $x = 0$: $ty = 2at + at^3$
$\therefore\ y = 2a + at^2$
As R lies on the parabola then $x^2 = 4y$
$\therefore\ x^2 = 4(2a + at^2)$
$\therefore\ x = \pm 2\sqrt{2+t^2}$
R is in the 2nd quadrant and so $x = -2\sqrt{2+t^2}$

Markers Comments

- This question was completed with a high level of accuracy.
- Students were penalised if they did not explain why the x-coordinate of R is negative

Question 7 (continued)

(b) (ii) Let M be the midpoint of RQ. Find the Cartesian equation of the locus of M. (Do NOT consider any possible domain restrictions of the locus) **2**

M is the midpoint of $\left(-2a\sqrt{t^2+2}, at^2+2a\right)$ and $\left(0, at^2+2a\right)$

$\therefore M\left(-a\sqrt{t^2+2}, at^2+2a\right)$

Let $x=-a\sqrt{t^2+2}$ and $y=at^2+2a$

$\therefore t^2=\dfrac{y-2a}{a}$

$$\begin{aligned} x^2 &= a^2\left(t^2+2\right) \\ &= a^2\left(\frac{y-2a}{a}+2\right) \\ &= a\left(y-2a+2a\right) \\ &= ay \end{aligned}$$

The locus of M is $x^2=ay$.

This is a parabola with the a quarter the focal length as $x^2 = 4ay$ but vertex at (0, 0).

Markers Comments

The question was generally well completed, though many students attempted to eliminate "a" instead of "t".

Question 8

(a) The polynomial $P(x) = x^3 + ax^2 + bx + 20$ has a factor of $x – 5$ and leaves a remainder of $–10$ when divided by $x – 3$. **3**

Find the values of a and b.

$P(5) = 0$: $125 + 25a + 5b + 20 = 0$
$\therefore 25a + 5b = –145$
$\therefore 5a + b = –29$ (1)

$P(3) = –10$ $27 + 9a + 3b + 20 = –10$
$\therefore 9a + 3b= –57$
$\therefore 3a + b = –19$ (2)

(1) – (2): $2a = –10$
$\therefore a = –5$
Substitute into (2): $–15 + b = –19$
$\therefore b = –4$

$\therefore a = –5, b = –4$

Marker's comment:

- Most students recognised that $P(5) = 0$ and $P(2) = –10$ and then generally were successful in finding a and b.
- There are students who cannot solve simultaneous equations without error.

(b) A curve has parametric equations $x = \frac{1}{t} - 1$ and $y = 2t + \frac{1}{t^2}$

(i) Find $\frac{dy}{dx}$ in terms of t. **2**

$x = t^{-1} - 1 \Rightarrow \frac{dx}{dt} = -t^{-2}$

$y = 2t + t^{-2} \Rightarrow \frac{dy}{dt} = 2 - 2t^{-3}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$
$$= \frac{2-2t^{-3}}{-t^{-2}}$$
$$= \frac{2t^3-2}{-t}$$
$$= -\frac{2t^3-2}{t}$$

Marker's comment:

Students who found the Cartesian equation to find $\frac{dy}{dx}$ found the process longer than those who were able to do it parametrically i.e. with the chain rule.

Question 8 (continued)

(b) (ii) Find the coordinates of any stationary points and determine their nature. **2**

Stationary points occur when $\frac{dy}{dx} = 0$ i.e. $-\frac{2t^3 - 2}{t} = 0$

$\therefore 2\left(t^3 - 1\right) = 0$

$\therefore t^3 = 1$

$\therefore t = 1$

Stationary point is at (0, 3)

$\frac{dy}{dx} = -\frac{2t^3 - 2}{t}$

$t = 0.5:$ $(1, 5)$ $\frac{dy}{dx} = \frac{7}{2}$

$t = 1:$ $(0, 3)$ $\frac{dy}{dx} = 0$

$t = 2:$ $\left(-\frac{1}{2}, 4\frac{1}{4}\right)$ $\frac{dy}{dx} = -7$

NB $\frac{dx}{dt} = -t^{-2}$ i.e. the *x*-coordinates are decreasing as *t* increases.

Using the parameter AND the coordinates then:

t	2	1	0
x	–0.5	0	1
$\frac{dy}{dx}$	–7	0	3.5

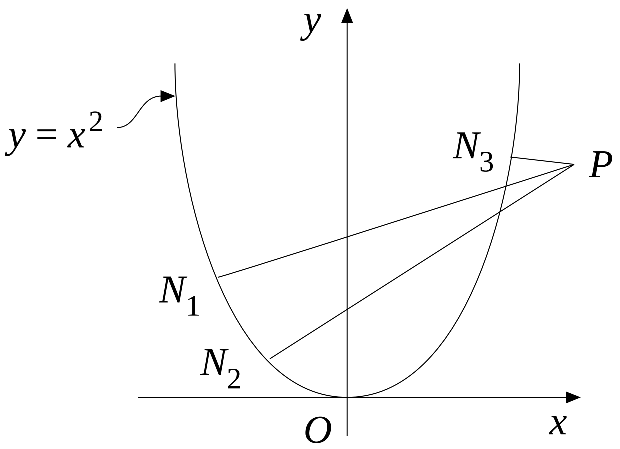
So (0, 3) is a minimum turning point.

Marker's comment:

- Many students did not make the correct conclusion that when $t < 1$ that $x > 1$ and vice versa. Hence many students incorrectly determined the nature of this stationary point. Students who did everything correctly but did not get the correct nature only lost 0.5 marks.
- Students who differentiated $\frac{dy}{dx}$ with respect to t to find the second derivative were not awarded a whole mark as it was mathematically incorrect.

Question 8 (continued)

(c) Consider the diagram below of the parabola $y = x^2$.
Some points (e.g P) lie on three distinct normals (PN_1, PN_2 and PN_3) to the parabola.



(i) Show that the equation of the normal to $y = x^2$ at the point (t, t^2) may be written as **2**

$$t^3 + \left(\frac{1-2y}{2}\right)t + \left(\frac{-x}{2}\right) = 0$$

$x = t$; $y = t^2$

$$\frac{dy}{dx} = 2x$$
$$= 2t$$

$\therefore$ the gradient of the normal is $-\dfrac{1}{2t}$

$$\therefore y - t^2 = -\frac{1}{2t}(x - t)$$
$$\therefore 2ty - 2t^3 = -x + t$$
$$\therefore 2t^3 + t - 2ty - x = 0$$
$$\therefore 2t^3 + t(1-2y) - x = 0$$
$$\therefore t^3 + t\left(\frac{1-2y}{2}\right) + \left(\frac{-x}{2}\right) = 0$$

Marker's comment:

- Successful students found that the gradient of the normal is $-\dfrac{1}{2t}$.
- Marks were deducted if students did not show enough working.

Question 8 (continued)

(c) (ii) For polynomials of the form $p(x) = x^3 + cx + d$, it is known that if the polynomial has 3 distinct roots then $27d^2 + 4c^3 < 0$ (Do NOT prove this.) **3**

Suppose that the normal to $y = x^2$ at three distinct points $N_1\left(t_1, t_1^{\ 2}\right)$, $N_2\left(t_2, t_2^{\ 2}\right)$ and $N_3\left(t_3, t_3^{\ 2}\right)$ all pass through $P\left(x_0, y_0\right)$.

Show that the coordinates of P satisfy $y_0 > 3\left(\frac{x_0}{4}\right)^{\frac{2}{3}} + \frac{1}{2}$

As there are 3 normals to $y = x^2$ that pass through $P\left(x_0, y_0\right)$, then the equation from (c) (i) i.e. $t^3 + \left(\frac{1-2y_0}{2}\right)t + \left(\frac{-x_0}{2}\right) = 0$ has 3 distinct solutions

If $c = \frac{1-2y_0}{2}$ and $d = \frac{-x_0}{2}$ then $27d^2 + 4c^3 < 0$

$$\therefore 27\left(-\frac{x_0}{2}\right)^2 + 4\left(\frac{1-2y_0}{2}\right)^3 < 0 \Rightarrow \frac{27x_0^{\ 2}}{4} + \frac{\left(1-2y_0\right)^3}{2} < 0$$

$$\therefore 27x_0^{\ 2} + 2\left(1-2y_0\right)^3 < 0 \Rightarrow 2\left(1-2y_0\right)^3 < -27x_0^{\ 2}$$

$$\therefore \left(1-2y_0\right)^3 < \frac{-27x_0^{\ 2}}{2}$$

$$\therefore 1-2y_0 < \left(\frac{-27x_0^{\ 2}}{2}\right)^{\frac{1}{3}} = -3\left(\frac{x_0^{\ 2}}{2}\right)^{\frac{1}{3}}$$

$$\therefore 1-2y_0 < -3\left(\frac{x_0^{\ 2}}{2}\right)^{\frac{1}{3}} = -3\left(\frac{x_0^{\frac{2}{3}}}{2^{\frac{1}{3}}}\right)$$

$$\therefore 2y_0 < 3\left(\frac{x_0^{\frac{2}{3}}}{2^{\frac{1}{3}}}\right) + 1$$

$$\therefore y_0 < 3\left(\frac{x_0^{\frac{2}{3}}}{2\times 2^{\frac{1}{3}}}\right) + \frac{1}{2} = 3\left(\frac{x_0}{4}\right)^{\frac{2}{3}} + \frac{1}{2}$$

Marker's comment:

- Students who expanded $\left(\frac{1-2y_0}{2}\right)^3$ struggled to find the required statement.
- Some students were able to substitute successfully (x_0, y_0) and were rewarded appropriately. As well as those students who correctly substituted the values for c and d.

End of solutions